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1982 J. Phys. A: Math. Gen. 15 605

(http://iopscience.iop.org/0305-4470/15/2/027)

J. Phys. A: Math. Gen. **15** (1982) 605–610. Printed in Great Britain

A note on percolation theory

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Received 6 May 1981

Abstract. In percolation theory the critical probability $P_c(G)$ of an infinite connected graph G is defined as the supremum of those values of the occupation probability for which only finite clusters occur.

An interesting question is the following: is each number between 0 and 1 the critical probability of some graph? It will be shown that the answer is positive.

A remarkable intermediate result is that for an important class of graphs the following holds: for each $p \geqslant P_c(G)$ there exists a subgraph of G with critical probability equal to p.

1. Introduction

Percolation theory, introduced by Broadbent and Hammersley in 1957, has become a fascinating field. It has many applications, especially in physics, where it gives insight in cooperative phenomena (e.g. spontaneous magnetism in a dilute ferromagnet) but also in biology (epidemics in a large orchard), geology and chemistry. Many such examples are described in Frisch and Hammersley (1963) and Shante and Kirkpatrick (1971).

Let G be an infinite non-oriented connected graph of which each vertex is the starting point of only a finite number of bonds. To this graph the following random mechanism is attached. Each bond is, independently of all other bonds, undammed with a fixed probability p and dammed with probability $1-p$. The terms dammed and undammed have been introduced by Broadbent and Hammersley for reasons of clearness (they describe the process as water, which is supplied to a given vertex and spreads from there through the undammed bonds). However, we prefer to use the terminology of Sykes and Essam (1964), and replace the words undammed and dammed by black and white respectively. Consequently, a walk is said to be black (white) if all its bonds are black (white). Further, the following definitions are important. For each vertex v, $P_n(p; v)$ denotes the probability that there are at least n vertices that can be reached from v via black walks. Obviously, $P_n(p; v)$ is decreasing in n and hence the limit $\lim_{n\to\infty} P_n(p; v)$ exists. This limit is denoted by $P_\infty(p; v)$.

The critical probability is defined as follows:

$$P_c(v) = \sup\{p | P_\infty(p; v) = 0\}. \tag{1}$$

Broadbent and Hammersley, who dealt with the more general case of partially oriented and not necessarily connected graphs, proved that if v_1 and v_2 are two vertices such that there exists a walk from v_1 to v_2 and also a walk *vice versa*, then $P_c(v_1) = P_c(v_2)$.

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Hence, since we consider only connected graphs, in our cases all vertices of a graph G have the same critical probability, which we denote by $P_c(G)$.

In another percolation model not the bonds but the vertices are randomly coloured. In this so-called site model we can give analogous definitions as for the bond model.

In general the critical probability for the bond process is not equal to that of the site process. Therefore, to make a distinction, we denote them by $P_c^{(b)}(G)$ and $P_c^{(s)}(G)$ respectively.

It can be shown (Fisher 1961) that the bond-percolation process on a graph G is equivalent with the site process on the so-called covering graph G^c of G, i.e.

$$P_c^{(b)}(G) = P_c^{(s)}(G^c). \tag{2}$$

We now turn to the central question of this paper: is any number p $(0 \leq p \leq 1)$ the critical probability of some graph G? It will be shown that this is indeed the case. From (2) it follows that it is sufficient to give a proof for the bond model. This proof is based on some well known results concerning the bond-percolation process on the square lattice, which we shall discuss in § 2.

2. The bond percolation process on the square lattice

The square lattice, denoted by S, consists of vertices $\{(n, m)|n, m \in \mathbb{Z}\}$, which all have one bond with each of their four neighbours.

The so-called dual lattice S^d of S is constructed as follows (see figure 1). Put one point in the centre of each face of S. These points $\{(n+\frac{1}{2}, m+\frac{1}{2})|n, m \in \mathbb{Z}\}$ form the vertex set of S^d. As we see, this graph S^d is again a square lattice, so that S and its dual are isomorphic. (This is generally not the case, e.g. the dual of the triangular lattice is the honeycomb lattice.) Therefore S is said to be self-dual.

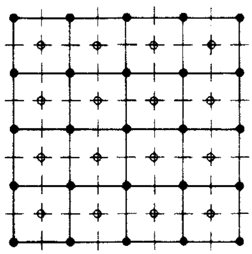

Figure 1. S and its dual S^d.

Each bond of S crosses exactly one bond of S^d so that the bond set of S is in 1–1 correspondence with that of S^d. So each colouring of the bonds of S induces a colouring of the bonds of S^d.

The following lemma is intuitively obvious. A proof is to be found in Whitney (1933).

Lemma 1. Each finite black cluster of S is surrounded by a white circuit of S^d. (This remains true after changing the terms black–white and/or the terms S–S^d.)

We shall now pay attention to the value of the critical probability $P_c^{(b)}(S)$ of S. Hammersley (1957), using the self-duality of S, proved that $e^{-v} \leqslant P_c^{(b)}(S) \leqslant 1-e^{-v}$, where v is the so-called connective constant of S. The lower bound has been improved by Harris (1960), who showed that $P_c^{(b)}(S) \geqslant \frac{1}{2}$. Although for a long time there had been many indications that in the last expression even equality holds (see e.g. Sykes and Essam, 1964), only recently a correct mathematical proof has been given, namely by Kesten (1980).

So we have

$$P_c^{(b)}(S) = \tfrac{1}{2}. \tag{3}$$

It has been proved by Harris (1960) that, for $p > P_c^{(b)}(S)$, almost surely (AS) there exists exactly one infinite black cluster. Hence, by (3) we have

Lemma 2. If $p > \frac{1}{2}$, then (AS) there is exactly one infinite black cluster in S.

Because this lemma plays an important role in the rest of the paper we let the proof (in a slightly different form) follow here. First note that the set of bonds of S is countable. When we denote the colour black by the number 1 and white by 0, then we can associate each bond b_i with a random variable x_i, which has the value 1 with probability p and the value 0 with probability $1-p$, and such that $\{x_i | i \in \mathbb{N}\}$ is a set of independent random variables. In these terms the event that there exists at least one infinite black cluster in S is a tail event of the sequence (x_i), $i \in \mathbb{N}$ (because, for each n, the existence of such a cluster does not depend on the colours of the bonds $b_0, b_1, \ldots, b_n$). Hence, by Kolmogorov's 0–1 law, the probability of this event is either 0 or 1. Now for $p > \frac{1}{2}$ this probability is, by (3), larger than 0 and therefore equal to 1.

The fact that, for $p > \frac{1}{2}$ (AS) not more than one infinite cluster exists can be seen as follows. Let v_1 and v_2 belong to the infinite black clusters C_1 and C_2 respectively. The probability of a bond to be white is $1-p$, which is smaller than $\frac{1}{2}$, so that (AS) all white clusters in S^d are finite.

But then it can be derived from lemma 1 that (AS) each finite set of vertices of S^d is surrounded by a black circuit in S, so (AS) there exists a black circuit in S which has both vertices v_1 and v_2 in its interior. It is obvious that this circuit connects C_1 and C_2, hence these clusters are one and the same.

3. A proof for the interval $[\frac{1}{2}, 1]$

In § 2 it has been stated that, for p larger than $\frac{1}{2}$, there exists (AS) exactly one infinite black cluster in S. It will appear that (AS) the critical probability $P_c^{(b)}$ of this cluster is equal to $\frac{1}{2}/p$. Then, by varying p in the interval $\langle \frac{1}{2}, 1]$, we can, for any value in $[\frac{1}{2}, 1\rangle$, 'create' a subgraph of S of which the critical probability is equal to that value. Subsequently, by a kind of trick, namely multiplication of the bonds of S, this result can be extended to the region $\langle 0, 1\rangle$. Next, only the trivial numbers 0 and 1 rest. As to the value 1, the easiest example of a graph with this critical probability is the linear chain consisting of vertices $v_1, v_2, v_3, \ldots$ and one bond between any pair (v_n, v_{n+1}). (In fact this graph can be considered as the section graph of S with vertex set $\{(x, 0) | x \in \mathbb{N}\}$.) Finally, the tree-like medium in figure 2 with vertex set $\{v_{n,m} n \geqslant 1, m \leqslant n!\}$, contains, for each k, the Bethe lattice of order k, so that its critical probability is, for each k, not larger than $1/k$ and hence equal to 0.

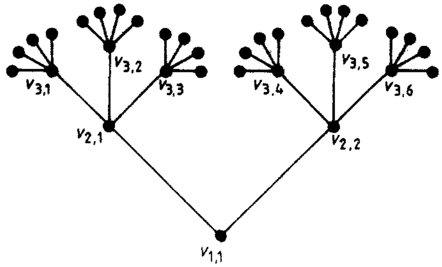

Figure 2. Example of a graph with critical probability 0.

We shall now prove the statement at the beginning of this section that, for $p > \frac{1}{2}$, the critical probability of the infinite black cluster is (AS) equal to $\frac{1}{2}/p$.

Let p_1 be a number in the interval $\langle\frac{1}{2}, 1]$ and let $\{b_i | i \in \mathbb{N}\}$ be the set of bonds of S. To this set corresponds a set $X = \{x_i | i \in \mathbb{N}\}$ of independent random variables, such that $\Pr\{x_i = 1\} = 1 - \Pr\{x_i = 0\} = p_1$.

The value 1 (0) of each random variable x_i corresponds with the state black (white) of its corresponding bond b_i. Further, let p_2 be any number in $[0, 1]$ and let $Y = \{y_i | i \in \mathbb{N}\}$ be a set of independent random variables such that $\Pr\{y_i = 1) = 1 - \Pr\{y_i = 0\} = p_2$ and X and Y are independent sets of random variables. Finally, define $Z = \{z_i | z_i = x_i y_i;\ i \in \mathbb{N}\}$.

The black subgraph corresponding to the x_i is called B', and the one corresponding to the z_i is called B''.

By the results in § 2 the following statements hold.

(i) Because $p_1 > \frac{1}{2}$, B' contains (AS) exactly one infinite cluster (see lemma 2), which we call C.

(ii) B'' is a subgraph of B' and contains (AS) no or exactly one infinite cluster. In the last case that cluster is a subgraph of C.

(iii) If $p_2 < \frac{1}{2}/p_1$, then, for all i, $\Pr\{z_i = 1\} = p_1 p_2 < \frac{1}{2}$ and hence (AS) B'' consists only of finite clusters.

(iv) On the other hand, if $p_2 > \frac{1}{2}/p_1$, then, for all i, $\Pr\{z_i = 1\} > \frac{1}{2}$ and hence (AS) B'' contains an infinite cluster, which, as stated in (ii), is a subgraph of C.

Now from the above it follows by definition that, (AS) the critical probability of C is indeed equal to $\frac{1}{2}/p_1$. Hence the class of those subgraphs of S which have critical probability $\frac{1}{2}/p_1$ is not empty. Next, by varying p_1 in the interval $\langle\frac{1}{2}, 1]$, and noting the example of a graph with critical probability 1 at the beginning of this section, we obtain the following theorem.

Theorem 1. Let p be a number in the interval $[\frac{1}{2}, 1]$. Then there exists a connected subgraph L of the square lattice with critical probability $P_c^{(b)}(L) = p$.

Remark. If G is a planar lattice, regularly built up of unit cells and possessing a pair of orthogonal symmetry axes, then it can be shown (see Fisher 1961), that $P_c^{(b)}(G) + P_c^{(b)}(G^d) \geq 1$, where G^d denotes the dual lattice of G. From this, by using the arguments in the proof of lemma 2, it can be proved that the following generalisation of that lemma holds: if $p > P_c^{(b)}(G)$ then there exists exactly one infinite black cluster in G. This, in its turn leads to a generalisation of theorem 1.

Each $p \geqslant P_c^{(b)}(G)$ is the critical probability of some subgraph of G.
An interesting question is whether this holds for all lattices.

4. Extension of the result in § 3 to the interval [0, 1]

Let S^n be the graph obtained by replacing each bond of S by n parallel bonds, $n \geqslant 1$ (see figure 3). For each colouring of the bonds of S^n a colouring of the bonds of S can be defined as follows: each bond of S is coloured black if at least one of the bonds of the corresponding n-tuple in S^n is black, otherwise it is coloured white. Hence, if p is the probability that a bond of S^n is coloured black, then the probability of a bond of S to be black is $1-(1-p)^n$. Further, note that there is an infinite black cluster in S if and only if there is one in S^n. From these reasonings it follows that $p \geqslant P_c^{(b)}(S^n)$ if and only if $1-(1-p)^n \geqslant P_c^{(b)}(S)$, which equals $\frac{1}{2}$, so that

$$P_c^{(b)}(S^n) = 1-[1-P_c^{(b)}(S)]^{1/n} = 1-(\tfrac{1}{2})^{1/n}. \tag{4}$$

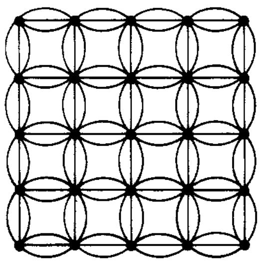

Figure 3. The lattice S^3.

Now we can apply the ideas of § 3 to S^n, which leads to the following theorem.

Theorem 2.

$$P_c^{(b)}(S^n) = 1-(\tfrac{1}{2})^{1/n}.$$

Further, if the probability p that a bond of S^n is black, is larger than $P_c^{(b)}(S^n)$, then (AS) there exists exactly one infinite black cluster in S^n and the critical probability of that cluster is equal to $P_c^{(b)}(S^n)/p$.

Now because $\lim_{n\to\infty} P_c^{(b)}(S^n) = 0$, the following theorem follows by varying n and p in theorem 2 (and again noting the example of the graph with critical probability 1 in § 3).

Theorem 3. For each p in the interval $\langle 0, 1]$ there exists, for a certain n, an infinite connected subgraph of S^n, of which the critical probability (bond case) is equal to p.

Theorem 3, together with the example of a graph with critical probability 0 (figure 2), completes the work.

Remark. If we do not want to deal with graphs with multiple bonds, like the S^n, we can handle them as follows. Define (instead of S^n) S^{n*} as the graph obtained by replacing each bond of S by an n-tuple of series of two bonds (figure 4). It is easily seen that the critical probability of S^{n^*} is equal to $(P_c^{(b)}(S^n))^{1/2}$ and a straightforward repeat of the arguments, earlier applied to S^n, leads to an analogue of theorem 3.

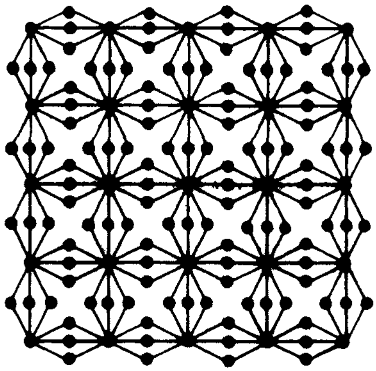

Figure 4. The lattice S^{3^*}

Acknowledgment

The author is indebted to Professor P W Kasteleyn for helpful comments.

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